

On a three-body confinement force in hadron spectroscopy

S. Pepin*

Max-Planck-Institut für Kernphysik, Postfach 103980, D-69029 Heidelberg, Germany

Fl. Stancu[†]

*University of Liège, Institute of Physics B5, Sart Tilman, B-4000 Liège 1, Belgium
ECT*, Strada delle Tabarelle, 286, I-38050 Villazzano, Trento, Italy*

(February 1, 2008)

Abstract

Recently it has been argued that a three-body colour confinement interaction can affect the stability condition of a three-quark system and the spectrum of a tetraquark described by any constituent quark model. Here we discuss the role of a three-body colour confinement interaction in a simple quark model and present some of its implications for the spectra of baryons, tetraquarks and six-quark systems.

I. INTRODUCTION

In the strong coupling limit the $SU_C(3)$ colour gauge group leads to a three-body colour confinement interaction in baryons [1]. In $SU(3)$ lattice QCD the static three-quark potential can be measured with the help of 3-quark Wilson loop operators (see for example [2–6]). In recent lattice calculations [4–6], the ground state potential of a three-quark system has been extracted as a sum of a two-body Coulomb plus a three-body interaction confinement. These studies lead however to rather contradictory results; one of them [5] gives support to the Y-type flux tube picture of Ref. [1], while the others favour the Δ ansatz (where the three-quark potential consists in a sum of two-body components).

The interaction potential obtained in these calculations corresponds in any case to the colourless ground state only and no information from lattice QCD about colour octets is available so far. In practice, for simplicity, in quark models as e.g. that of [1] the confinement is treated approximately as a two-body colour operator. This can be expressed in terms of the quadratic (Casimir) invariant operator of $SU(3)$.

This three-body colour confinement interaction should not be confused with the three-body force [7,8] associated to the instanton 't Hooft's interaction, which in the nonrelativistic

*e-mail address: pepin@daniel.mpi-hd.mpg.de

[†]e-mail address: fstancu@ulg.ac.be

limit contains a colour operator similar to the one introduced below, but is of short range, in contradistinction to confinement forces. The instanton-induced three-body force cancels for three quarks in a colour singlet state and is moreover only effective if the three particles are in a flavour singlet state. Another short-range three-body force has also been introduced in [9] on a purely phenomenological ground. This force has a simple scalar structure and accounts for a better description of the Roper resonance.

Based on the algebraic argument that $SU_C(3)$ is an exact symmetry of QCD, which implies that any quark model Hamiltonian inspired by QCD can be written in terms of $SU(3)$ invariant operators, a three-quark potential that depends on the cubic invariant operator of $SU(3)$ has recently [10] been added to the usual two-body confinement. Its implications on the spectrum of ordinary q^3 and exotic $q^2\bar{q}^2$ hadrons have been considered. In particular turning on a three-body force with an appropriate strength denoted by c , it was shown that: 1) the three q^3 colour states, namely **1**, **8** and **10** appear in the correct order, **1** being the lowest one, as it should be, 2) in the $q^2\bar{q}^2$ system the three-body interaction brings distinct contributions to the two possible colour singlet states, by enhancing the binding in one and diminishing it in the other, depending on the sign of c .

In this study we rederive some of the relations found in Ref. [10] and discuss explicitly the role of confining three-body forces in q^3 , $q^2\bar{q}^2$ and q^6 systems. In the simple framework of a harmonic confinement we show that in the q^3 system there is a competition between the three-body force and the kinetic energy in rising the energy of the colour octet states. By introducing singlet-singlet and octet-octet $q\bar{q}$ coupled pairs we show that the range of values of the strength of the 3-body force giving a correct spectrum for a q^3 system also favourably affects the spectrum of a tetraquark $q^2\bar{q}^2$. The results on the role of a 3-body confining force in a q^6 system, relevant for the NN problem are entirely new.

II. THE THREE-BODY FORCE FOR BARYONS

In this section we recall and discuss the findings of Ref. [10] in relation with a three-body interaction of type

$$V_{3b} = V_{ijk} = \mathcal{V}_{ijk} \mathcal{C}_{ijk} , \quad (1)$$

with

$$\mathcal{V}_{ijk} = \frac{1}{2} c m \omega^2 [(r_i - r_j)^2 + (r_j - r_k)^2 + (r_k - r_i)^2] , \quad (2)$$

where c is a strength parameter and \mathcal{C}_{ijk} a colour operator of type

$$\mathcal{C}_{ijk} = d^{abc} F_i^a F_j^b F_k^c , \quad (3)$$

where $F_i^a = \frac{1}{2} \lambda_i^a$ is the colour charge operator of the quark i and d^{abc} some real constants, symmetric under any permutation of indices and defined by the anticommutator of the Gell-Mann matrices λ^a as

$$\{\lambda^a, \lambda^b\} = 2d^{abc} \lambda^c . \quad (4)$$

These constants satisfy the following orthogonality relation

$$d^{abc} d^{abe} = \frac{5}{3} \delta_{ce} \quad (5)$$

The operator (3) can be expressed in terms of the two independent invariant operators of SU(3) as [10] (for a proof see Appendix A)

$$d^{abc} F_i^a F_j^b F_k^c = \frac{1}{6} \left[C_{i+j+k}^{(3)} - \frac{5}{2} C_{i+j+k}^{(2)} + \frac{20}{3} \right] \quad (6)$$

where we slightly changed the notation of [10], by writing $C^{(2)}$ instead of $C^{(1)}$ for the quadratic invariant and $C^{(3)}$ instead of $C^{(2)}$ for the cubic invariant. For a given irrep of SU(3) labelled by $(\lambda\mu)$, the eigenvalues of these invariants are

$$\langle C^{(2)} \rangle = \frac{1}{3} (\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) \quad (7)$$

and (see for example Refs. [11] or [12])

$$\langle C^{(3)} \rangle = \frac{1}{18} (\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3) . \quad (8)$$

Then for a q^3 system the expectation values of (6) are $\frac{10}{9}$, $-\frac{5}{36}$, $\frac{1}{9}$ for a singlet $(\lambda\mu) = (00)$, octet $(\lambda\mu) = (11)$ and decuplet $(\lambda\mu) = (30)$ states respectively. These are the coefficients appearing in the last term of Eq. (14) below.

Turning on a 2-body confining interaction, which ensures stability for a $q\bar{q}$ pair and adding the 3-body confining interaction (1)-(3) of strength c relative to the 2-body one, in Ref. [10] it was found that the spectrum of a q^3 system is correctly described provided

$$-\frac{3}{2} < c < \frac{2}{5} \quad (9)$$

The closer c is to the lower limit, the larger is the gap between the colour octet and singlet states. To see this, let us consider the Hamiltonian

$$H = T + V_{2b} + V_{3b} \quad (10)$$

where T is the kinetic energy and V_{2b} a 2-body confinement interaction of the form

$$V_{2b} = \sum_{i < j} V_{ij} \left(c_1 + \frac{4}{3} + F_i^a F_j^a \right) \quad (11)$$

containing an arbitrary constant c_1 which we set equal to 1 as in Ref. [10] and take

$$V_{ij} = \frac{1}{2} m\omega^2 (r_i - r_j)^2 \quad (12)$$

V_{3b} is the 3-body confinement interaction of Eqs. (1)-(3). Performing integration in the colour space and expressing H in terms of the internal coordinates $\vec{\rho} = (\vec{r}_1 - \vec{r}_2)/\sqrt{2}$ and $\vec{\lambda} = (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)/\sqrt{6}$, we have:

$$H = 3m - \frac{\hbar^2}{2m}(\nabla_\rho^2 + \nabla_\lambda^2) + \frac{3}{2}m\omega^2\chi_i(\rho^2 + \lambda^2) \quad (13)$$

with

$$\chi_i = \begin{cases} \frac{5}{3} + \frac{10}{9}c & i=1 \text{ (singlet)} \\ \frac{13}{6} - \frac{5}{36}c & i=8 \text{ (octet)} \\ \frac{8}{3} + \frac{1}{9}c & i=10 \text{ (decuplet)} \end{cases} \quad (14)$$

In the expressions of χ_i ($i = 1, 8$ or 10), the first and second terms stem from the colour part of V_{2b} and V_{3b} respectively.

We search now for solutions of H . In order to satisfy the Pauli principle, in the lowest colour-singlet state the quarks should be in a s^3 configuration. This implies that the orbital wave function is symmetric and one can take:

$$\phi_{00} = \frac{1}{\pi^{3/2}b^3} \exp [-(\rho^2 + \lambda^2)/(2b^2)] \quad (15)$$

with b a variational parameter.

The expectation value of H is then:

$$E_1 = 3m + \frac{3\hbar^2}{2mb^2} + \frac{9}{2}m\omega^2b^2\chi_1 \quad (16)$$

The minimization with respect to b^2 gives:

$$b_1^2 = \frac{b_0^2}{\sqrt{3\chi_1}} \quad (17)$$

with $b_0^2 = \frac{\hbar}{m\omega}$. The energy of the singlet state is thus:

$$E_1 = 3m + \frac{3\hbar^2}{mb_1^2} = 3m + 3\hbar\omega\sqrt{3\chi_1} \quad (18)$$

The colour-octet **8** must be combined with an s^2p configuration in order to satisfy the Pauli principle. Indeed, the spin-flavour part being totally symmetric, as for the nucleon ground state, the orbital-colour part must be antisymmetric, i.e. it is of the form:

$$\Psi_8 = \frac{1}{\sqrt{2}}(\phi_{10}^\rho C^\lambda - \phi_{10}^\lambda C^\rho) \quad (19)$$

where

$$\phi_{10}^\rho = \left(\frac{2}{\pi^3 b^8}\right)^{1/2} \rho_z \exp [-(\rho^2 + \lambda^2)/(2b^2)] \quad (20)$$

$$\phi_{10}^\lambda = \left(\frac{2}{\pi^3 b^8}\right)^{1/2} \lambda_z \exp [-(\rho^2 + \lambda^2)/(2b^2)] \quad (21)$$

are the mixed orbital symmetry states with one unit of angular excitation and C^ρ , C^λ their colour counter-parts. The corresponding eigenvalue of H is:

$$E_8 = 3m + \frac{2\hbar^2}{mb^2} + 6m\omega^2 b^2 \chi_8 \quad (22)$$

and the minimization with respect to b^2 gives:

$$b_8^2 = \frac{b_0^2}{\sqrt{3\chi_8}} \quad (23)$$

which leads to

$$E_8 = 3m + \frac{4\hbar^2}{mb_8^2} = 3m + 4\hbar\omega\sqrt{3\chi_8} \quad (24)$$

Accordingly the gap ΔE between the octet and singlet states is:

$$\Delta E = \hbar\omega(4\sqrt{3\chi_8} - 3\sqrt{3\chi_1}) \quad (25)$$

The largest value of ΔE corresponds to $\chi_1 = 0$, i.e. to $c = -1.5$, in which case $\Delta E \simeq 10.7\hbar\omega$. For $c = 0$ (no three-body force) we would have $\Delta E \simeq 3.5\hbar\omega$. This means that the gap is enlarged by a negative c and triples for the limiting value $c = -1.5$.

The colour state **10** requires an antisymmetric orbital state in order to satisfy the Pauli principle. Its form is (see e.g. Ref. [12], chap. 10):

$$\phi_{10}^A = \left(\frac{1}{2\pi^3 b_{10}}\right)^{1/2} (\rho_+ \lambda_- - \rho_- \lambda_+) \exp[-(\rho^2 + \lambda^2)/(2b^2)] \quad (26)$$

with $\rho_\pm = \rho_x \pm i\rho_y$, etc. The subscript 10 means total $L=1$, $M=0$ as above. This is the only value of L allowed by an antisymmetric state built from the configuration sp^2 . Proceeding in a similar way as for the two previous cases, one gets:

$$b_{10}^2 = \frac{b_0^2}{\sqrt{3\chi_{10}}} \quad (27)$$

and hence

$$E_{10} = 3m + 5\hbar\omega\sqrt{3\chi_{10}} \quad (28)$$

The gap between the decuplet and the octet state is thus:

$$E_{10} - E_8 = \hbar\omega(5\sqrt{3\chi_{10}} - 4\sqrt{3\chi_8}) \quad (29)$$

For $c = -1.5$, one has $E_{10} - E_8 \simeq 3\hbar\omega$, i.e. this state is located above the octet, as expected, with quite a large gap for a limiting value of c .

Let us now evaluate the octet-singlet gap (25) by taking $c = -1.43$ as in [10]. There is of course some arbitrariness in choosing $\hbar\omega$. One can take for example $b = 0.437$ fm and $m = 0.340$ GeV, as typical values for a quark model (see e.g. Ref. [15]), which give $\hbar\omega = 0.6$

GeV. This implies a gap $\Delta E \approx 5.5$ GeV. For $c = 0$ (no three-body force) one would have $\Delta E = 3.5\hbar\omega \approx 2.1$ GeV.

Before ending this section we should note that there is some arbitrariness in fixing the lower limit of the range of c as given by the inequality (9). This limit is related to the choice of the arbitrary constant c_1 which has been set equal to 1 in Eq. (11). But taking for example $c_1 = 4/3$, which is another good choice in constituent quark models, one gets $\chi_1 = 2 + \frac{10}{9}c$. The stability condition for the singlet would then gives

$$-\frac{9}{5} < c \quad (30)$$

i. e. a different lower limit, slightly more favourable than the one of inequality (9) because with $c = -9/5$ and the new expressions for χ_1, χ_8 one gets $\Delta E \simeq 11.5\hbar\omega$.

III. THE THREE-BODY FORCE FOR TETRAQUARKS

If F^a is the colour charge operator of a quark, for an antiquark we must have

$$\overline{F}^a = -\frac{1}{2}\lambda^{a*} \quad (31)$$

in order that \overline{F}^a ($a = 1, 2, \dots, 8$) satisfy the Lie algebra too. Then one can write the three-body interaction acting in the $q^2\overline{q}$ subsystem as

$$\overline{\mathcal{C}}_{ijk} = -d^{abc} F_i^a F_j^b \overline{F}_k^c \quad (32)$$

and the three-body interaction acting in the $q\overline{q}^2$ subsystem as

$$\overline{\mathcal{C}}_{ijk} = d^{abc} F_i^a \overline{F}_j^b \overline{F}_k^c \quad (33)$$

As a remark, the three-body interaction in an antibaryon should be

$$\overline{\mathcal{C}}_{ijk} = -d^{abc} \overline{F}_i^a \overline{F}_j^b \overline{F}_k^c. \quad (34)$$

In Ref. [10] the operator (32) is given in terms of SU(3) invariants as

$$\overline{\mathcal{C}}_{ijk} = -\frac{1}{6}[C_{i+j+\overline{k}}^{(3)} - \frac{5}{2}C_{i+j}^{(2)} + \frac{50}{9}] \quad (35)$$

where $C_{i+j+\overline{k}}^{(3)}$ acts on the $q^2\overline{q}$ subsystem but the Casimir operator acts only on the subsystem of $i+j$ quarks (see Appendix A). If the quark subsystem is in a symmetric state it gives rise to a $q^2\overline{q}$ $[211]_C$ state called s and if it is in an antisymmetric state to a $[211]_C$ state called a . Both these states have $(\lambda\mu) = (10)$, which according to (8) gives $\langle C_{i+j+\overline{k}}^{(3)} \rangle = \frac{10}{9}$. But for the subsystem of the $i+j$ quarks only, the SU(3) representations are different. One has $(\lambda\mu) = (20)$ for the s state and $(\lambda\mu) = (01)$ for the a state. Then the expectation value of the operator (35) is $-\frac{5}{18}$ for s and $\frac{5}{9}$ for a , consistent with Table II of [10].

The operator (33) acting on a $q\bar{q}^2$ subsystem can be brought to a form similar to (35). This is

$$\bar{\mathcal{C}}_{ijk} = \frac{1}{6}[C_{i+\bar{j}+\bar{k}}^{(3)} + \frac{5}{2}C_{\bar{j}+\bar{k}}^{(2)} - \frac{50}{9}] \quad (36)$$

The difference with respect to a $q^2\bar{q}$ subsystem is now that one has to calculate the expectation value of $C_{i+\bar{j}+\bar{k}}^{(3)}$ for a $[221]_C$ colour state for which $(\lambda\mu) = (01)$ so that one has now $\langle C_{i+\bar{j}+\bar{k}}^{(3)} \rangle = -\frac{10}{9}$. The subsystem of antiquarks gives for $C^{(2)}$ the same value as that for the quarks so that the operators (32) and (33) have the same expectation value which leads to Eqs. (26) and (27) of Ref. [10]

$$V_s = \frac{5}{18} c m\omega^2 (r_{12}^2 + r_{13}^2 + r_{14}^2 + r_{23}^2 + r_{24}^2 + r_{34}^2) \quad (37)$$

$$V_a = -\frac{5}{9} c m\omega^2 (r_{12}^2 + r_{13}^2 + r_{14}^2 + r_{23}^2 + r_{24}^2 + r_{34}^2) \quad (38)$$

One can see that the contribution of the fourth particle is also included in these equation. To understand this one can for example add another antiquark to $q^2\bar{q}$. This leads to the singlet colour state $[222]$ appearing from the direct product $[211] \times [11]$. By construction this singlet has an intermediate coupling both between quarks and antiquarks. The two quarks couple either to a $\bar{3}$ or a 6 state and the antiquarks to 3 or a $\bar{6}$ state. If the particles 1 and 2 are quarks and 3 and 4 are antiquarks, V_a and V_s are the three-body contribution to the colourless states denoted by $|\bar{3}_{12}3_{34}\rangle$ and $|6_{12}\bar{6}_{34}\rangle$ respectively.

For a negative c , as required for baryons described by a pure constituent quark model (no gluon components in the wave function), the mass of the $|6_{12}\bar{6}_{34}\rangle$ state is reduced and the mass of $|\bar{3}_{12}3_{34}\rangle$ is enhanced by a three-body force. The situation is opposite for a positive c . In [10], V_s was associated to the unobserved sextet-sextet state, which would mean that a positive c is preferable. The conflict can be solved by noting that the relevant states in the present problem are in fact linear combinations of $|\bar{3}_{12}3_{34}\rangle$ and $|6_{12}\bar{6}_{34}\rangle$. Such states are important asymptotically and they are defined by the transformations (see e.g. Ref. [14])

$$|1_{13}1_{24}\rangle = \sqrt{\frac{1}{3}}|\bar{3}_{12}3_{34}\rangle + \sqrt{\frac{2}{3}}|6_{12}\bar{6}_{34}\rangle \quad , \quad (39)$$

$$|8_{13}8_{24}\rangle = -\sqrt{\frac{2}{3}}|\bar{3}_{12}3_{34}\rangle + \sqrt{\frac{1}{3}}|6_{12}\bar{6}_{34}\rangle \quad . \quad (40)$$

In these states the intermediate coupling in the colour space takes place between a quark q and an antiquark \bar{q} . This gives colour singlet $q\bar{q}$ pairs in Eq. (39) and colour octet ones in Eq. (40). Asymptotically the energy of $|8_{13}8_{24}\rangle$ must become large, as such a state is not expected to be seen. Using the transformations (39) and (40) one obtains the contribution of the three-body interaction in a $q^2\bar{q}^2$ system as

$$\langle 1_{13}1_{24}|\bar{\mathcal{C}}_{123}|1_{13}1_{24}\rangle \propto \left[\frac{1}{3}\left(-\frac{5}{9}\right) + \frac{2}{3}\frac{5}{18}\right] c = 0 \quad (41)$$

and

$$\langle 8_{13}8_{24}|\bar{\mathcal{C}}_{123}|8_{13}8_{24}\rangle \propto [\frac{2}{3}(-\frac{5}{9}) + \frac{1}{3}\frac{5}{18}]c = -\frac{5}{18}c \quad (42)$$

which shows that with a negative c one raises the expectation value of the octet-octet above the singlet-singlet state, more than with $c = 0$. This implies that the coupling between octet-octet and singlet-singlet states due to a hyperfine splitting will be diminished, which amounts to make a ground state tetraquark less stable. This seems to be consistent with the experimental observation that no stable tetraquark system has been seen so far.

IV. THE NN INTERACTION

The short-range NN interaction can be studied as a q^6 problem. First we give a simplified discussion by considering that the six quarks are in a totally symmetric orbital state $[6]_O$. In such a case the spin-flavour part of the wave function has a $[33]_{FS}$ symmetry which combined with the colour symmetry $[222]_C$ state leads to a totally antisymmetric state. The latter is a superposition of five colour components given by the five following Young tableaux:

$$\psi_1 = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}, \quad \psi_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}, \quad \psi_3 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}, \quad \psi_4 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}, \quad \psi_5 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \quad (43)$$

Below we give some details of our calculations of the three-body matrix elements for a $6q$ system. In a state of orbital symmetry $[6]_O$, i.e. of configuration s^6 , all orbital matrix elements are equal so one has only to calculate the colour matrix element:

$$\frac{1}{5} \sum_{i < j < k}^6 \sum_{a,b,c}^5 \sum_{n=1}^5 \langle \psi_n | d^{abc} F_i^a F_j^b F_k^c | \psi_n \rangle \quad (44)$$

where the factor $1/5$ comes from the normalization of the total wave function (see e.g. [12], chapter 10). In Appendix B we have explicitly proved that:

$$\sum_{a,b,c} \langle \psi_2 | d^{abc} F_1^a F_2^b F_3^c | \psi_2 \rangle = -5/36 \quad (45)$$

We get the same result for ψ_3, ψ_4 and ψ_5 but not for ψ_1 , for which one has:

$$\sum_{a,b,c} \langle \psi_1 | d^{abc} F_1^a F_2^b F_3^c | \psi_1 \rangle = 10/9 \quad (46)$$

i.e., the result for the singlet (123), as expected. As shown in Table 1, the matrix elements $\langle \psi_n | d^{abc} F_i^a F_j^b F_k^c | \psi_n \rangle$ differ for different (ijk) but the sum over the five states doesn't depend on the choice of (ijk) ; the table has been calculated for a given value of the colour-indices $(abc) = (146)$ but the conclusion is also true for the other values of the colour indices. Then the calculation of the matrix element (44) reduces to the calculation of the matrix element of the three-quarks (123) which has to be multiplied by $C_6^3 = 20$.

Thus we obtain

$$\begin{aligned}
\frac{1}{5} \sum_{i < j < k} \sum_{a,b,c} \sum_{n=1}^5 < \psi_n | d^{abc} F_i^a F_j^b F_k^c | \psi_n > &= \frac{20}{5} \sum_{a,b,c} \sum_{n=1}^5 < \psi_n | d^{abc} F_1^a F_2^b F_3^c | \psi_n > \\
&= 4 \left(\sum_{a,b,c} < \psi_1 | d^{abc} F_1^a F_2^b F_3^c | \psi_1 > + 4 \sum_{a,b,c} < \psi_2 | d^{abc} F_1^a F_2^b F_3^c | \psi_2 > \right) \quad (47) \\
&= 4(10/9 - 20/36) = 20/9
\end{aligned}$$

Therefore we can see that in the case of a three-body confining force the value of the matrix element of a six-quark system in the symmetry state $[6]_O$ is equal to two times the value for a single baryon. This situation is similar to the two-body force where the expectation value of a 2-body operator $V_{2b} \propto \sum_{i < j} F_i^a F_j^a$ in the symmetry state $[6]_O$ is equal to - 4 i.e. two times the value of a single baryon. Let us define an adiabatic NN potential as the difference between the interaction Hamiltonian at zero separation distance and at infinity, i.e.

$$V_{NN} = H(0) - H(\infty) \quad (48)$$

In this difference only the kinetic energy survives if the NN system is in the state $[6]_O$. The contribution of the confinement due both to two- and three-body forces cancels out because

$$V^{conf}(0) = V^{conf}(\infty) = V_{2b} + V_{3b} = -4 + 20/9c \quad (49)$$

The two-body confinement force has been discussed for example in Ref. [15] where the Hamiltonian also contains a hyperfine interaction.

However the physical NN state is a combination of three symmetry states given by [16]

$$|NN\rangle = \sqrt{\frac{1}{9}} |[6]_O[33]_{FS}\rangle + \sqrt{\frac{4}{9}} |[42]_O[33]_{FS}\rangle - \sqrt{\frac{4}{9}} |[42]_O[51]_{FS}\rangle \quad (50)$$

For $SI=(01)$ or (10) , one should also consider the physical $\Delta\Delta$ state:

$$|\Delta\Delta\rangle = \sqrt{\frac{4}{45}} |[6]_O[33]_{FS}\rangle + \sqrt{\frac{16}{45}} |[42]_O[33]_{FS}\rangle + \sqrt{\frac{25}{45}} |[42]_O[51]_{FS}\rangle \quad (51)$$

The unphysical colour octet-octet (CC) state has the form [16]

$$|CC\rangle = \sqrt{\frac{4}{5}} |[6]_O[33]_{FS}\rangle - \sqrt{\frac{1}{5}} |[42]_O[33]_{FS}\rangle \quad (52)$$

By using 3-body fractional parentage coefficients (cfp) (given in Appendix C), we calculated the expectation value of the 3-body potential acting on the symmetry states $[42]_O[33]_{FS}\rangle$ and $[42]_O[51]_{FS}\rangle$. For $[6]_O[33]_{FS}\rangle$, the result is straightforward as shown above. In short, we have found the following expectation values:

$$< [6]_O[33]_{FS} | V_{3b} | [6]_O[33]_{FS} > = \frac{20}{9}c \quad (53)$$

$$< [42]_O[33]_{FS} | V_{3b} | [42]_O[33]_{FS} > = \frac{1}{9}c \quad (54)$$

$$< [42]_O[51]_{FS} | V_{3b} | [42]_O[51]_{FS} > = \frac{1}{9}c \quad (55)$$

Note that the $[[42]_O[33]_{FS} >$ and $[[42]_O[51]_{FS} >$ states have the same expectation value, consistent with the fact that V_{3b} is spin-isospin independent. Using the transformations (50-52) from the symmetry states to the physical states NN , $\Delta\Delta$ and the hidden colour CC state, we obtain the following matrix for V_{3b} :

$$\begin{array}{c|ccc}
 & NN & \Delta\Delta & CC \\
 \hline
 NN & \frac{28}{81}c & \frac{38\sqrt{5}}{405}c & \frac{38\sqrt{5}}{135}c \\
 \Delta\Delta & \frac{38\sqrt{5}}{405}c & \frac{121}{405}c & \frac{76}{135}c \\
 CC & \frac{38\sqrt{5}}{135}c & \frac{76}{135}c & \frac{9}{5}c
 \end{array} \tag{56}$$

The eigenvalues of this matrix are $E_1 = c/9$, $E_2 = c/9$ and $E_3 = 20c/9$. This shows that the effect of the 3-body colour confinement on NN and $\Delta\Delta$ is identical and rather small as compared to that on CC . In particular for a negative value of c , the spectrum of NN , $\Delta\Delta$ and CC lowers and shrinks. For a positive c , the situation is the other way round. This means that, for $c < 0$, V_{3b} itself brings some attraction and implies a stronger coupling of CC to NN and $\Delta\Delta$ due to a hyperfine interaction. This will lead to a reduced hard core repulsion in the NN potential.

V. CONCLUSIONS

We discussed the role of a schematic three-body confinement force in the spectra of $3q$, $q^2\bar{q}^2$ and q^6 systems. We found that a three-body confinement interaction with a negative strength c has the following effects: 1) it increases the gap between the physical colour singlet state and the unphysical coloured states in baryons; 2) it raises the expectation value of $q\bar{q}$ pairs in colour octet-octet states with respect to singlet-singlet states in tetraquark systems, which will lead to a smaller binding in a ground state tetraquark when a hyperfine interaction is included and 3) it increases the coupling between physical states and CC states in q^6 systems. While, in the first two cases, the gap between the physical colour-singlet state and the non-physical coloured states is increased, the opposite is true for the $6q$ system. The larger coupling between physical and CC states induced by the 3-body interaction has both negative and positive consequences: it will reinforce the undesirable Van der Waals forces but, on the other hand, it brings more attraction into the NN potential. This may be a desired feature for quark models which give a too strong hard core repulsion.

A three-body force with a negative strength c will have just opposite effects than the ones mentioned above.

Our results are valid for any quark model, irrespective of the hyperfine interaction. It would be useful to extend this study to a more realistic confinement interaction.

Note also that our conclusions for $q^2\bar{q}^2$ and q^6 correspond to a zero separation between the interacting clusters (2 mesons and 2 baryons respectively). It may be possible that the

contribution of the confinement interaction changes with the separation distance. This is the aim of a further study.

APPENDIX A:

In this Appendix we first prove Eq. (6). In order to avoid any confusion, we fix the indices (ijk) of (6) to be (123). We rewrite Eq. (3) as

$$\begin{aligned} d^{abc} F_1^a F_2^b F_3^c &= \frac{1}{6} \left[\sum_{i,j,k}^3 d^{abc} F_i^a F_j^b F_k^c \right. \\ &\quad \left. - 3 \sum_{i,j}^3 d^{abc} F_i^a F_i^b F_j^c + 2 \sum_i^3 d^{abc} F_i^a F_i^b F_i^c \right] \end{aligned} \quad (\text{A1})$$

where the second sum in the right-hand side compensates for the extra terms contained in the first sum, but as we extract too many we add the third term for satisfying the equality correctly. The first term is precisely the cubic invariant operator acting on the three-quark system $C_{1+2+3}^{(3)}$ and the last term is 2×3 times the cubic invariant operator $C_1^{(3)}$ acting on a quark. The latter is replaced by its eigenvalue $\frac{10}{9}$ so we get

$$d^{abc} F_1^a F_2^b F_3^c = \frac{1}{6} \left[C_{1+2+3}^{(3)} - 3 \sum_{i,j}^3 d^{abc} F_i^a F_i^b F_j^c + \frac{20}{3} \right] \quad (\text{A2})$$

Due to the fact that the constants d^{abc} are symmetric under the permutation of indices we can modify the second term as

$$\begin{aligned} d^{abc} F_1^a F_2^b F_3^c &= \frac{1}{6} \left[C_{1+2+3}^{(3)} \right. \\ &\quad \left. - \frac{3}{2} \sum_{i,j}^3 d^{abc} \{F_i^a, F_i^b\} F_j^c + \frac{20}{3} \right] \end{aligned} \quad (\text{A3})$$

and simplify it by using the anticommutator (4) in the form

$$\{F_i^a, F_i^b\} = d^{abc} F_i^c \quad (\text{A4})$$

and the orthogonality relation (5). This leads to

$$d^{abc} F_1^a F_2^b F_3^c = \frac{1}{6} \left[C_{1+2+3}^{(3)} - \frac{5}{2} C_{1+2+3}^{(2)} + \frac{20}{3} \right] \quad (\text{A5})$$

i.e. Eq. (6).

Next we prove Eq. (35). One can rewrite the operator (32) as

$$\begin{aligned}
-d^{abc} F_1^a F_2^b \bar{F}_3^c &= -\frac{1}{6} [d^{abc} (F_1^a + F_2^a + \bar{F}_3^a)(F_1^b + F_2^b + \bar{F}_3^b)(F_1^c + F_2^c + \bar{F}_3^c) \\
&\quad -3 \sum_{i,j}^2 d^{abc} F_i^a F_i^b F_j^c + 2 \sum_i^2 d^{abc} F_i^a F_i^b F_i^c \\
&\quad -3 \sum_i^2 d^{abc} F_i^a F_i^b \bar{F}_3^c \\
&\quad -3 \sum_i^2 d^{abc} F_i^a \bar{F}_3^b \bar{F}_3^c \\
&\quad -d^{abc} \bar{F}_3^a \bar{F}_3^b \bar{F}_3^c] \tag{A6}
\end{aligned}$$

The operator in the first term of the right-hand side is the $C^{(3)}$ invariant associated with the whole system formed of the quarks 1 and 2 and the antiquark $\bar{3}$. As in Eq. (A1), the extra-terms introduced by this operator must be compensated in order to recover the left-hand side. This is the role of the other terms.

The first term can be replaced by $C_{1+2+\bar{3}}^{(3)}$ and the second by $5/2 C_{1+2}^{(2)}$ where the factor $5/2$ has the same explanation as in Eq. (A5). The third term is 2×2 the invariant operator for a single quark $C_1^{(3)} = 10/9$. The last term contains only antiquark charge operators and is thus identical to $C_{\bar{3}}^{(3)} = -10/9$. The sum over the constant terms gives

$$4C_q^{(3)} - C_{\bar{q}}^{(3)} = 50/9 \tag{A7}$$

where $q = 1$ or 2 and $\bar{q} = \bar{3}$, so we have

$$-d^{abc} F_1^a F_2^b \bar{F}_3^c = -\frac{1}{6} [C_{1+2+\bar{3}}^{(3)} - \frac{5}{2} C_{1+2}^{(2)} - 3 \sum_i^2 d^{abc} F_i^a F_i^b \bar{F}_3^c - 3 \sum_i^2 d^{abc} F_i^a \bar{F}_3^b \bar{F}_3^c + \frac{50}{9}] \tag{A8}$$

Due to the fact that d^{abc} are symmetric under permutation of indices a, b and c we can again use the identity

$$d^{abc} F_i^a F_i^b \bar{F}_3^c = \frac{1}{2} d^{abc} \{F_i^a, F_i^b\} \bar{F}_3^c \tag{A9}$$

and

$$d^{abc} F_i^a \bar{F}_3^b \bar{F}_3^c = \frac{1}{2} d^{abc} F_i^a \{\bar{F}_3^b, \bar{F}_3^c\} \tag{A10}$$

From (4) and (31) it follows that

$$\{\bar{F}_i^a, \bar{F}_i^b\} = -d^{abc} \bar{F}_i^c. \tag{A11}$$

From (A4) and (A11) it follows that the third and fourth term compensate each other and we get

$$\bar{\mathcal{C}}_{123} = -\frac{1}{6} [C_{1+2+\bar{3}}^{(3)} - \frac{5}{2} C_{1+2}^{(2)} + \frac{50}{9}] \tag{A12}$$

i.e. equation (35).

APPENDIX B:

In this Appendix we give the details leading to Eq. (45). We first present the explicit expressions for the colour wave functions corresponding to the Young tableaux of Eq. (43). In the state ψ_1 , the sets of particles (123) and (456) are both in a totally antisymmetric state. Therefore, one can write:

$$\psi_1 = \frac{1}{6} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(3) & b(3) & g(3) \end{vmatrix} \begin{vmatrix} r(4) & b(4) & g(4) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \quad (\text{B1})$$

where r, b, g denotes the different quark colours. By applying the permutation (34) to ψ_1 , one gets [12] :

$$(34)\psi_1 = \frac{1}{3}\psi_1 + \frac{2\sqrt{2}}{3}\psi_2 \quad (\text{B2})$$

from where one obtains :

$$\psi_2 = \frac{3}{2\sqrt{2}} \left[(34)\psi_1 - \frac{1}{3}\psi_1 \right] \quad (\text{B3})$$

$$= \frac{1}{4\sqrt{2}} \left(\begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(4) & b(4) & g(4) \end{vmatrix} \begin{vmatrix} r(3) & b(3) & g(3) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} - \frac{1}{3} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(3) & b(3) & g(3) \end{vmatrix} \begin{vmatrix} r(4) & b(4) & g(4) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \right) \quad (\text{B4})$$

The states ψ_3, ψ_4 and ψ_5 can be obtained by a similar procedure. They read:

$$\begin{aligned} \psi_3 = \frac{1}{2\sqrt{6}} & \left(\begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(5) & b(5) & g(5) \end{vmatrix} \begin{vmatrix} r(3) & b(3) & g(3) \\ r(4) & b(4) & g(4) \\ r(6) & b(6) & g(6) \end{vmatrix} - \frac{1}{2} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(4) & b(4) & g(4) \end{vmatrix} \begin{vmatrix} r(3) & b(3) & g(3) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \right. \\ & \left. + \frac{1}{2} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(3) & b(3) & g(3) \end{vmatrix} \begin{vmatrix} r(4) & b(4) & g(4) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \right) \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} \psi_4 = \frac{1}{2\sqrt{6}} & \left(\begin{vmatrix} r(1) & b(1) & g(1) \\ r(3) & b(3) & g(3) \\ r(4) & b(4) & g(4) \end{vmatrix} \begin{vmatrix} r(2) & b(2) & g(2) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} - \frac{1}{2} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(4) & b(4) & g(4) \end{vmatrix} \begin{vmatrix} r(3) & b(3) & g(3) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \right. \\ & \left. + \frac{1}{2} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(3) & b(3) & g(3) \end{vmatrix} \begin{vmatrix} r(4) & b(4) & g(4) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \right) \end{aligned} \quad (\text{B6})$$

$$\begin{aligned}
\psi_5 = & \frac{1}{3\sqrt{2}} \left(\begin{vmatrix} r(1) & b(1) & g(1) \\ r(3) & b(3) & g(3) \\ r(5) & b(5) & g(5) \end{vmatrix} \begin{vmatrix} r(2) & b(2) & g(2) \\ r(4) & b(4) & g(4) \\ r(6) & b(6) & g(6) \end{vmatrix} - \frac{1}{2} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(5) & b(5) & g(5) \end{vmatrix} \begin{vmatrix} r(3) & b(3) & g(3) \\ r(4) & b(4) & g(4) \\ r(6) & b(6) & g(6) \end{vmatrix} \right. \\
& - \frac{1}{2} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(3) & b(3) & g(3) \\ r(4) & b(4) & g(4) \end{vmatrix} \begin{vmatrix} r(2) & b(2) & g(2) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} + \frac{1}{4} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(4) & b(4) & g(4) \end{vmatrix} \begin{vmatrix} r(3) & b(3) & g(3) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \\
& \left. - \frac{3}{4} \begin{vmatrix} r(1) & b(1) & g(1) \\ r(2) & b(2) & g(2) \\ r(3) & b(3) & g(3) \end{vmatrix} \begin{vmatrix} r(4) & b(4) & g(4) \\ r(5) & b(5) & g(5) \\ r(6) & b(6) & g(6) \end{vmatrix} \right)
\end{aligned} \tag{B7}$$

We want now to calculate the expression $\sum_{a,b,c} \langle \psi_2 | d^{abc} F_1^a F_2^b F_3^c | \psi_2 \rangle$. This can be done explicitly using Eq. (B4). The non vanishing terms in this sum are given in Table II. Taking into account the multiplicity of each term, one can directly check that the final result is -5/36. Similar calculations can be done for the functions ψ_3 , ψ_4 and ψ_5 leading to the same answer.

APPENDIX C:

In this appendix, we give the values of the 3-body coefficients of fractional parentage (cfp) necessary to calculate the expectation value of the 3-body potential and we sketch the method to determine them.

The state $|[42]_O[33]_{FS} \rangle$ can be decomposed as:

$$\begin{aligned}
|[42]_O[33]_{FS} \rangle = & \sqrt{\frac{1}{5}} \begin{vmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{vmatrix}_{OC} \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{vmatrix}_{FS} - \sqrt{\frac{1}{5}} \begin{vmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{vmatrix}_{OC} \begin{vmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{vmatrix}_{FS} + \sqrt{\frac{1}{5}} \begin{vmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{vmatrix}_{OC} \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix}_{FS} \\
& - \sqrt{\frac{1}{5}} \begin{vmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{vmatrix}_{OC} \begin{vmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix}_{FS} - \sqrt{\frac{1}{5}} \begin{vmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{vmatrix}_{OC} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}_{FS}
\end{aligned} \tag{C1}$$

One has to determine the 3-body cfp associated to the decomposition of the OC part of the wave function into its orbital and colour parts, for example:

$$\begin{vmatrix} & \\ & 4 \\ 5 & 6 \end{vmatrix}_{OC} \rightarrow \begin{vmatrix} & & 5 & 6 \\ & 4 & & \end{vmatrix}_O \begin{vmatrix} & \\ & 4 \\ 5 & 6 \end{vmatrix}_C \tag{C2}$$

To determine the 3-body cfp we need to write the Clebsch-Gordan (CG) coefficients of S_6 specifying the place of the last three particles (pqr), where p,q,r represent the row in the Young tableau where the particles 6, 5 and 4 are located. The position of the remaining particles is denoted shortly by y . By using the factorization properties of the CG [17], one gets the following relations :

$$\begin{aligned}
S([f']p'q'r'y'[f'']p''q''r''y''|[f]pqry) &= K([f']p'[f'']p''|[f]p) \\
&\times S([f'_p]q'r'y'[f''_p]q''r''y''|[f_p]qry) \\
&= K([f']p'[f'']p''|[f]p) K([f'_p]q'[f''_p]q''|[f_p]q) \\
&\times S([f'_{p'q'}]r'y'[f''_{p''q''}]r''y''|[f_{pq}]ry) \\
&= K([f']p'[f'']p''|[f]p) K([f'_p]q'[f''_p]q''|[f_p]q) \\
&\times K([f'_{p'q'}]r'[f''_{p''q''}]r''|[f_{pq}]r) \\
&\times S([f'_{p'q'r'}]y'[f''_{p''q''r''}]y''|[f_{pqr}]y)
\end{aligned} \tag{C3}$$

where the quantities K are isoscalar factors and S are CG coefficients. In particular the last factor is the CG of S_3 . We use the same notations as in Ref. [17] : $[f_p]$ corresponds to the partition of S_5 obtained after removal of the particle 6, $[f_{pq}]$ to the partition of S_4 obtained after removal of the particle 5, etc.

The 3-body cfp is defined as:

$$\begin{aligned}
K_3([f']p'q'r'y'[f'']p''q''r''|[f]pqr) &= K([f']p'[f'']p''|[f]p) K([f'_p]q'[f''_p]q''|[f_p]q) \\
&\times K([f'_{p'q'}]r'[f''_{p''q''}]r''|[f_{pq}]r)
\end{aligned} \tag{C4}$$

The values of the K_3 can then be calculated by using the corresponding tables of Ref. [17]. They are listed in Tables III and IV. They give respectively the cfp relevant for the decomposition of the $[222]_{OC}$ and $[21^4]_{OC}$ state. In the calculation of the expectation values, the CG of S_3 are not necessary, as they are added up in the orthogonality relation.

Acknowledgements. We acknowledge warm hospitality and living support from ECT*, Trento where this collaboration has started. We are most grateful to David Brink for stimulating discussions at an early stage of this study and for simplifying the proof of Eq. (6) given in Appendix A. Useful discussions with Jean-Marc Richard and Mitja Rosina are also gratefully acknowledged.

REFERENCES

- [1] J. Carlson, J. Kogut and V. R. Pandharipande, Phys. Rev. **D27** (1983) 233.
- [2] H. Dosch and V. Mueller, Nucl. Phys **B116** (1976) 470
- [3] D. Gromes, Phys. Rep. **200** (1991) 127
- [4] G. Bali, Phys. Rep. **343** (2001) 1
- [5] T. T. Takahashi, H. Matsufuru, Y. Nemoto and H. Suganuma, Phys. Rev. Lett. **86** (2001) 18
- [6] C. Alexandrou, Ph. de Forcrand and A. Tsapalis, hep-lat/0107006.
- [7] A. E. Dorokhov and N. I. Kochelev, preprint JINR-E2-86-847, 1986
- [8] M. Oka and S. Takeuchi, Nucl. Phys. **A524** (1991) 649; S. Takeuchi and M. Oka, Phys. Rev. Lett. **66** (1991) 1271.
- [9] B. Desplanques, C. Gignoux, B. Silvestre-Brac, P. Gonzalez, J. Navarro and S. Noguera, Z. Phys. **A343** (1992) 331.
- [10] V. Dmitrasinovic, Phys. Lett **B499** (2001) 135.
- [11] J. P. Elliott and P. G. Dawber, *Symmetry in Physics* MacMillan, London, 1979
- [12] Fl. Stancu, *Group Theory in Subnuclear Physics* Oxford University Press, Oxford, 1996
- [13] J. Linde, T. Ohlsson and H. Snellman Phys. Rev. **D57**, 452 (1998)
- [14] D. M. Brink and Fl. Stancu, Phys. Rev. **D49**, 4665 (1994)
- [15] Fl. Stancu, S. Pepin and L. Ya. Glozman, Phys. Rev. **C56**, 2779 (1997)
- [16] M. Harvey, Nucl. Phys. **A352** (1981) 301; *ibid.* **A481** (1988) 834
- [17] Fl. Stancu and S. Pepin, Few-Body Systems **26** (1999) 113.

TABLES

TABLE I. Examples of the matrix elements $\langle \psi_n | d^{abc} F_i^a F_j^b F_k^c | \psi_n \rangle$ for a few values of the indices (ijk) at fixed $(abc) = (146)$. The successive columns correspond to the states ψ_1, \dots, ψ_5 ; the last column gives the sum over the five states.

(ijk)	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	$\sum_{n=1}^5$
(123)	1/48	-1/384	-1/384	-1/384	-1/384	1/96
(145)	0	-1/768	1/768	1/768	7/768	1/96
(124)	0	7/384	-1/384	-1/384	-1/384	1/96

TABLE II. Values of the non-vanishing matrix elements $\langle \psi_2 | d^{abc} F_1^a F_2^b F_3^c | \psi_2 \rangle$. The first column gives the colour indices (abc) ; the second the corresponding constant d^{abc} and the third column the value of the matrix element.

(abc)	d^{abc}	$\langle \psi_2 d^{abc} F_1^a F_2^b F_3^c \psi_2 \rangle$
118	$\sqrt{3}/3$	-1/288
146	1/2	-1/384
157	1/2	-1/384
228	$\sqrt{3}/3$	-1/288
247	-1/2	-1/384
256	1/2	-1/384
338	$\sqrt{3}/3$	-1/288
344	1/2	-1/384
355	1/2	-1/384
366	-1/2	-1/384
377	-1/2	-1/384
448	$-\sqrt{3}/6$	-1/1152
558	$-\sqrt{3}/6$	-1/1152
668	$-\sqrt{3}/6$	-1/1152
778	$-\sqrt{3}/6$	-1/1152
888	$-\sqrt{3}/3$	-1/288

TABLE III. The 3-body cfp $K_3([42]p'q'r'[222]p''q''r''|[222]pqr)$. The rows correspond to $p'q'r'$ and the columns to $p''q''r''$. The value of pqr is given in the upper-left corner of the table.

$pqr = 332$	332	323	321
221	$-\sqrt{2/12}$		
212		$\sqrt{10/108}$	
211		$-\sqrt{10/108}$	$\sqrt{10/108}$
122		$-\sqrt{5/108}$	
121		$\sqrt{5/108}$	$-\sqrt{5/108}$
112	$-\sqrt{5/12}$		

$pqr = 323$	332	323	321
221		$\sqrt{4/54}$	
212	$\sqrt{10/108}$	$-\sqrt{5/162}$	
211	$-\sqrt{10/108}$	$-\sqrt{20/162}$	$-\sqrt{5/162}$
122	$-\sqrt{5/108}$	$-\sqrt{10/162}$	
121	$\sqrt{5/108}$	$-\sqrt{40/162}$	$-\sqrt{10/162}$
112		$-\sqrt{5/108}$	$\sqrt{5/108}$

$pqr = 321$	332	323	321
221			$\sqrt{4/54}$
212			$\sqrt{20/162}$
211	$-\sqrt{20/108}$	$\sqrt{10/162}$	
122			$\sqrt{40/162}$
121	$\sqrt{5/54}$	$\sqrt{20/162}$	
112		$-\sqrt{5/54}$	

TABLE IV. Same as Table III but for $K_3([42]p'q'r'[222]p''q''r''|[21^4]pqr)$.

$pqr = 543$	332	323	321
221		$\sqrt{5/27}$	
212	$-\sqrt{4/27}$	$-\sqrt{1/81}$	
211	$\sqrt{4/27}$	$-\sqrt{4/81}$	$-\sqrt{1/81}$
122	$\sqrt{2/27}$	$-\sqrt{2/81}$	
121	$-\sqrt{2/27}$	$-\sqrt{8/81}$	$-\sqrt{2/81}$
112		$\sqrt{2/27}$	$-\sqrt{2/27}$
$pqr = 541$	332	323	321
221			$\sqrt{5/27}$
212			$\sqrt{4/81}$
211	$\sqrt{8/27}$	$\sqrt{2/81}$	
122			$\sqrt{8/81}$
121	$-\sqrt{4/27}$	$\sqrt{4/81}$	
112		$\sqrt{4/27}$	
$pqr = 154$	332	323	321
122			$\sqrt{1/5}$
121		$-\sqrt{2/5}$	
112	$\sqrt{2/5}$		
$pqr = 514$	332	323	321
212			$-\sqrt{5/27}$
211		$\sqrt{10/27}$	
122			$-\sqrt{8/135}$
121		$\sqrt{16/135}$	
112	$\sqrt{4/15}$		